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INTERMEDIATE COUPLING CALCULATIONS IN THE UNIFIED NUCLEAR MODEL

BY

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The intermediate coupling treatment of the unified nuclear model, discussed by A. Bohr and B. Mottelson, is extended to a study of the nuclear level structure for a single $j = 5/2$ particle coupled to the nuclear surface oscillations of the quadrupole type. Magnetic moments and quadrupole moments for the nuclear ground states for $j = 5/2$ and $j = 7/2$ configurations are also considered. Wave functions are used, including all unperturbed states with up to three phonons, and the energies and moments are calculated as a function of the coupling strength. We should need the inclusion of states with still more phonons to make detailed contact with the strong coupling region. However, even for the intermediate coupling strength for which the present results are valid, various features of the strongly coupled system are beginning to develop.

I. Introduction.

In the unified description of nuclear dynamics[1,2], the nucleus is considered as a shell structure capable of performing collective oscillations. The state of the system is thus described in terms of individual-particle and collective degress of freedom. The former represent the most loosely bound particles, while the latter represent the bulk of the nucleons which cannot be individually excited at moderate nuclear excitation energies. The most important of the collective types of the motion are oscillations in the nuclear shape which resemble surface oscillations.

The collective motion involves variations in the nuclear field and is, therefore, coupled to the motion of the individual nucleons. The properties of the system depend essentially on the strength of this coupling, which again depends on the particle configuration and on the nuclear deformability.

The coupled system possesses simple solutions in the limit of weak and strong coupling. In the former case, the particle and the collective types of motion are approximately independent and the effect of the coupling can be treated as a small perturbation.

[1] A. Bohr, Dan. Mat. Fys. Medd. **26**, no. 14, 1952; in the text quoted as A.
[2] A. Bohr and B. Mottelson, Dan. Mat. Fys. Medd. **27**, no. 16, 1953; in the text quoted as B.-M.

In the latter case, the system bears analogies to molecular structures; the nucleus acquires a large deformation and the stationary states can be characterized by the motion of the particles with respect to the deformed nucleus and the vibration and rotation of the structure as a whole.

A weak coupling situation is expected in the vicinity of major closed shells, where the high stability of the spherical nuclear shape makes the coupling relatively ineffective. With the addition of particles, the coupling increases and, in regions far removed from closed shells, the strong coupling situation appears to be rather accurately realized, as evidenced in particular by the nuclear rotational spectra (cf., e. g., B.-M. § VIc).

In many nuclei, however, neither the weak nor the strong coupling solutions are adequate, and it is necessary to develop methods to analyze the properties of the system also in the intermediate coupling region. Some calculations to this effect have been performed (B.-M. § IIb.iii), using the representation of uncoupled motion similar to that used by FOLDY and MILFORD[1], and diagonalizing the Hamiltonian, including states containing up to a certain number of quanta of the surface oscillations. The method is somewhat similar to the Tamm-Dancoff[2,3] method of field theory. In the present paper, we apply this method to further studies of nuclear properties in the intermediate coupling region.

Calculations of this type become rather complex when the number of unperturbed states included in the wave function becomes too great, and we therefore restrict ourselves to the simplest type of system, that of a single particle with a constant angular momentum j coupled to the nuclear surface oscillations of lowest order (the quadrupole oscillations). In the description of nuclear states, it will often be necessary to consider many-particle configurations and also to take into account that the interaction with the surface oscillations may couple together particle states with different values of j. It is expected, however, that the simplified system considered will contain many of the characteristic features of the intermediate coupling situation and illustrate the gradual transition from weak to strong coupling.

[1] L. L. FOLDY and F. J. MILFORD, Phys. Rev. 80, 751, 1950.
[2] I. TAMM, J. Phys. U.S.S.R. 9, 449. 1945.
[3] S. M. DANCOFF, Phys. Rev. 78, 382, 1950.

In Section II, we give the equation of motion and derive the general formula for the evaluation of matrix elements for the particle-surface interaction. In Sections III, IV, and V, we consider level structure, magnetic moments and quadrupole moments, respectively, for a number of configurations.

II. Equation of Motion; Method of Treatment.

We consider the dynamical system consisting of a single particle with angular momentum j coupled to the quadrupole oscillations of the nuclear surface. The total Hamiltonian of the system has the form (we follow the notation of A. and B.-M.)

$$H = H_s(\alpha_{2\mu}) + H_p(x) + H_{\text{int}}(\alpha_{2\mu}, x) \cdots, \tag{1}$$

where the surface energy H_s is equivalent to that of a system of harmonic oscillators and is given by

$$H_s(\alpha_{2\mu}) = \sum_{\mu} \left\{ \frac{1}{2} B_2 \left| \dot{\alpha}_{2\mu} \right|^2 + \frac{1}{2} C_2 \left| \alpha_{2\mu} \right|^2 \right\} \cdots \cdots \tag{2}$$

in terms of the mass parameter B_2 and nuclear deformability C_2 (defined in A. and B.-M.). The frequency of the surface oscillations is given by

$$\omega = \sqrt{\frac{C_2}{B_2}} \cdots \cdots . \tag{3}$$

The particle Hamiltonian H_p is a constant, since the particle is assumed to remain in the same orbits so that only the degree of freedom associated with the orientation of its state is taken into account.

The interaction energy H_{int} represents the coupling of the particle to the nuclear deformation and to first order in $\alpha_{2\mu}$ takes the form

$$H_{\text{int}}(\alpha_{2\mu}, x) = -k \sum_{\mu} \alpha_{2\mu} Y_{2\mu}(\theta, \varphi) \cdots \cdots, \tag{4}$$

where θ, φ are the polar angles of the particle and k is the coupling constant.

The effect of the particle-surface interaction may be conveniently characterized by the dimensionless coupling parameter given by

$$x = \sqrt{\frac{5}{4\pi}} \cdot \frac{1}{\sqrt{4j}} \cdot \frac{k}{\sqrt{\hbar\omega C}} \cdots\cdots \tag{5}$$

For $x\sqrt{j} \ll 1$, the coupling is relatively ineffective and can be treated by perturbation method; for $x \gg 1$ the strong coupling treatment again provides a simple solution of the coupled motion.

In the present intermediate coupling treatment, we shall use the representation of uncoupled motion and thus write the wave function

$$\Psi = \sum_{NR} |j; NR; IM\rangle \langle j; NR; IM|\rangle \cdots \cdots, \tag{6}$$

where the state of the surface is characterized by the number of phonons N, each having an anugular momentum of two units, and by R, the total angular momentum of the surface. I and M denote the total angular momentum of the nucleus and its Z component, respectively.

In order to evaluate the matrix elements of H_{int}, it is convenient to write the surface variable $\alpha_{2\mu}$ in terms of creation and destruction operators b_μ^* and b_μ, respectively. The expression (4) for H_{int} then becomes

$$H_{\text{int}}(\alpha_{2\mu}) = -k \sum_{\mu} \sqrt{\frac{\hbar\omega}{2C}} \cdot (b_\mu + (-1)^\mu b_{-\mu}^*) Y_{2\mu}(\theta, \varphi). \tag{7}$$

Using the decomposition

$$|j: NR; IM\rangle = \sum_{m\mu'} |jm\rangle |NR\mu'\rangle (jRm\mu' | jRIM) \cdots \cdots, \tag{8}$$

where the first two terms represent the particle state vector and the surface state vector, respectively, and where $(jRm\mu' | jRIM)$ is the Clebsch-Gordan coefficient[1] for the composition of angular momenta; we obtain for $N' > N$

$$\left.\begin{aligned} &\langle j; NR; IM | H_{\text{int}} | j; N'R'; IM\rangle \\ &= -k\sqrt{\frac{\hbar\omega}{2C}} \sum_{mm'\mu\mu'\mu''} \langle jm | Y_{2\mu} | jm'\rangle \cdot \langle NR\mu' | b_\mu | N'R'\mu''\rangle \\ &\times (jRm\mu' | jRIM) \cdot (jR'm'\mu'' | jR'IM) \cdots \cdots \end{aligned}\right\} \tag{9}$$

[1] E. U. Condon and G. H. Shortley. The Theory of Atomic Spectra, Cambridge University Press, 1935.

The general matrix elements of $\langle Y_{2\mu} \rangle$ have the form

$$\langle lsjm \,|\, Y_{2\mu} \,|\, l'sj'm' \rangle = \langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle \cdot (j' 2\, m' m - m' \,|\, j' 2 jm)^* \cdots \cdots, \quad (10)$$

where l is the orbital angular momentum and s the spin of the particle.

The function $\langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle$ is evaluated and is given by

$$\left.\begin{array}{l} \langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle \\ = (-1)^{j + l' - \frac{1}{2}} \cdot \sqrt{\dfrac{5}{4\pi}} \cdot \sqrt{(2l'+1)(2j'+1)}\, W(j'l'jl \,|\, {}^1\!/_2 2) \cdot (l' 200 \,|\, l' 2 l 0) \cdots, \end{array}\right\} \quad (11)$$

where $W(j'l'jl \,|\, {}^1\!/_2 2)$ is a Racah coefficient, the values of which have been tabulated[1]. It can be shown that $\langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle$ depends only on j and j' and not on l and l', provided the combining states have the same parity.

Similarly, we can write out the matrix elements of $\langle b_\mu \rangle$ in the form

$$\langle NRM_R \,|\, b_\mu \,|\, N'R'M'_R \rangle = \langle NR \,\|\, b \,\|\, N'R' \rangle \cdot (R 2 M_R \mu \,|\, R 2 R' M_R + \mu) \cdots \cdots (12)$$

In evaluating the function $\langle NR \,\|\, b \,\|\, N'R' \rangle$, it is convenient to write the phonon states in terms of Boson creation operators acting on the ground state of the nuclear surface. As an illustration, the two-phonon state with angular momentum R can be written as

$$|\, 2Rm \rangle = \frac{1}{A} \sum_{\mu\mu'} (22\mu\mu' \,|\, 22Rm) \cdot b^*_\mu b^*_{\mu'} \,|\, 0 \rangle \cdots \cdots \cdots \cdots, \quad (13)$$

where A is the normalization factor of the state vector and is obtained by computing the absolute value of $\sum_{\mu\mu'} (22\mu\mu' \,|\, 22Rm) \times b^*_\mu b^*_{\mu'} \,|\, 0 \rangle$. The values of $\langle NR \,\|\, b \,\|\, N'R' \rangle$ have been evaluated for all states involving up to three phonons and are given in Table I.

By means of (10) and (12), one obtains from (9) an expression for the matrix elements of H_{int} which involves products of four

* Note that our notation $\langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle$ is not the same as RACAH's. The relation is $\sqrt{2j+1} \cdot \langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle_{\text{our}} = (lsj \,\|\, Y_2 \,\|\, l'sj')_{\text{Racah's}}$ (Cf. Phys. Rev. 62, 438, 1942).

[1] L. C. Biedenharn, J. M. Blatt, and M. E. Rose, Rev. Mod. Phys. 24, 249, 1952.

TABLE I.

The values of $\langle NR \| b \| N'R' \rangle$ factors.

$\langle 00 \| b \| 12 \rangle = 1$	$\langle 22 \| b \| 33 \rangle = \sqrt{\frac{15}{7}}$
$\langle 12 \| b \| 20 \rangle = \sqrt{2}$	$\langle 22 \| b \| 34 \rangle = \sqrt{\frac{11}{7}}$
$\langle 12 \| b \| 22 \rangle = \sqrt{2}$	$\langle 24 \| b \| 32 \rangle = \sqrt{\frac{36}{35}}$
$\langle 12 \| b \| 24 \rangle = \sqrt{2}$	$\langle 24 \| b \| 33 \rangle = -\sqrt{\frac{6}{7}}$
$\langle 20 \| b \| 32 \rangle = \sqrt{\frac{7}{5}}$	$\langle 24 \| b \| 34 \rangle = \sqrt{\frac{10}{7}}$
$\langle 22 \| b \| 30 \rangle = \sqrt{3}$	$\langle 24 \| b \| 36 \rangle = \sqrt{3}$
$\langle 22 \| b \| 32 \rangle = \sqrt{\frac{4}{7}}$	

Clebsch-Gordan coefficients. Summing over the magnetic quantum numbers we finally obtain

$$\left.\begin{aligned} &\langle j; NR; IM \,|\, H_{\text{int}} \,|\, j; N'R'; IM \rangle \\ &= (-1)^{R+I+1-j} \cdot k \cdot \sqrt{\frac{\hbar\omega}{2C}} \cdot \sqrt{(2j+1)(2R'+1)}\, W(R'Rjj \,|\, 2I) \\ &\quad \times \langle lsj \| Y_2 \| lsj \rangle \cdot \langle NR \| b \| N'R' \rangle \cdots\cdots \end{aligned}\right\} \quad (14)$$

We now obtain an approximate solution of the equation of motion by including in the wave function (6) only states involving up to a certain number of phonons and by diagonalizing this restricted system. Assuming first a value for the eigenvalue, one may solve the linear equations in the amplitudes $\langle j; NR; IM | \rangle$ and, in turn, obtain an improved energy value. Such an iteration method was found to converge rapidly.

III. Level Structure.

We have studied the level structure for a particle with $j = 5/2$ coupled to the nuclear surface. The case $j = 5/2$ was chosen because, with increasing j, the number of phonons occurring for a given coupling strength increases. On the other hand, for $j = 1/2$, the coupling vanishes and, in case of $j = 3/2$, there is no regular strong coupling solution with which to compare.

We have calculated the energies as a function of the coupling parameter x of the two lowest $I = 5/2$ states of which the lower represents the ground state of the system; besides these,

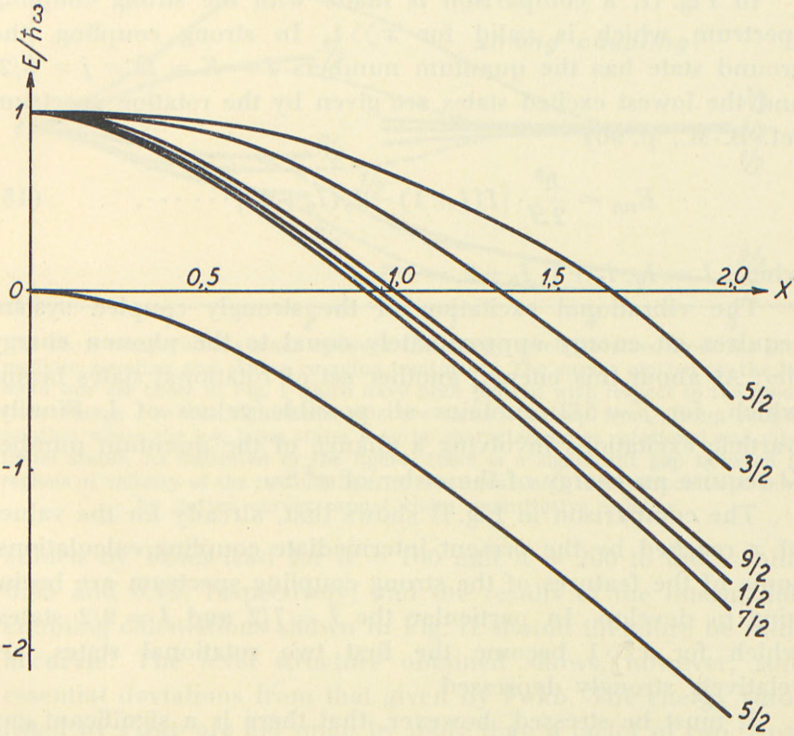

Fig. I. Level structure of a single $j = 5/2$ particle coupled to the nuclear surface oscillations. Energies of the two lowest states of $I = 5/2$ in units of $\hbar\omega$ have been plotted as a function of the coupling parameter $x = \sqrt{\frac{5}{4\pi}} \cdot \sqrt{\frac{1}{4j}} \cdot \frac{k}{\sqrt{\hbar\omega C}}$. The lower curve represents the ground state of the system. The other four curves represent the energies of the lowest states for types $I = 1/2$, $3/2$, $7/2$, and $9/2$. Wave-functions including all unperturbed states with up to three phonons have been used in the calculations. The results obtained are believed to be reliable for $x < 1 \cdot 5$.

the energies of the lowest states of types $I = 1/2$, $3/2$, $7/2$, and $9/2$ have also been evaluated. In the calculations we have included all the unperturbed states including up to three phonons. The results obtained are shown in Fig. I.

The region of validity of the present calculations can be seen from the magnitude of the amplitudes of the three phonon states and it is found that significant effects of states with a higher number of phonons are to be expected for x larger than 1.5. It may be added that a comparison with the results obtained by only including states up to two phonons shows that the latter approximation is only valid for $x < 1$.

In Fig. II, a comparison is made with the strong coupling spectrum which is valid for $x \gg 1$. In strong coupling, the ground state has the quantum numbers $I = K = \Omega = j = 5/2$, and the lowest excited states are given by the rotation spectrum (cf. B.-M., p. 96)

$$E_{\mathrm{rot}} = \frac{\hbar^2}{2\mathfrak{J}} \cdot \left[I(I+1) - I_0(I_0+1) \right] \cdots\cdots, \qquad (15)$$

where $I = I_0, I_0 + 1, I_0 + 2, \cdots\cdots$

The vibrational excitation of the strongly coupled system requires an energy approximately equal to the phonon energy $\hbar\omega$. At about this energy, another set of rotational states begins which, for $j = 5/2$, contains all possible values of I. Finally, particle excitations involving a change of the quantum number Ω require an energy of the order of $x^2 \hbar\omega$.

The comparison in Fig. II shows that, already for the values of x reached by the present intermediate coupling calculations, some of the features of the strong coupling spectrum are beginning to develop. In particular the $I = 7/2$ and $I = 9/2$ states, which for $x \gg 1$ become the first two rotational states, are relatively strongly depressed.

It must be stressed, however, that there is a significant gap between the regions of validity of the two methods of treatment, and the interpolations, shown in the figure by dotted curves, cannot therefore claim quantitative validity. In order to carry the present calculations to such large values of x that a detailed contact with the strong coupling spectrum can be made, it would be necessary to include in the wave function many more phonons, which would make calculations with the present techniques rather impracticable.

The present results can be compared with those obtained by FORD[1] who has calculated the level structure for a single particle ($j = 5/2$) configuration in the coupled system by using the strong coupling approximation. The nuclear parameters as-

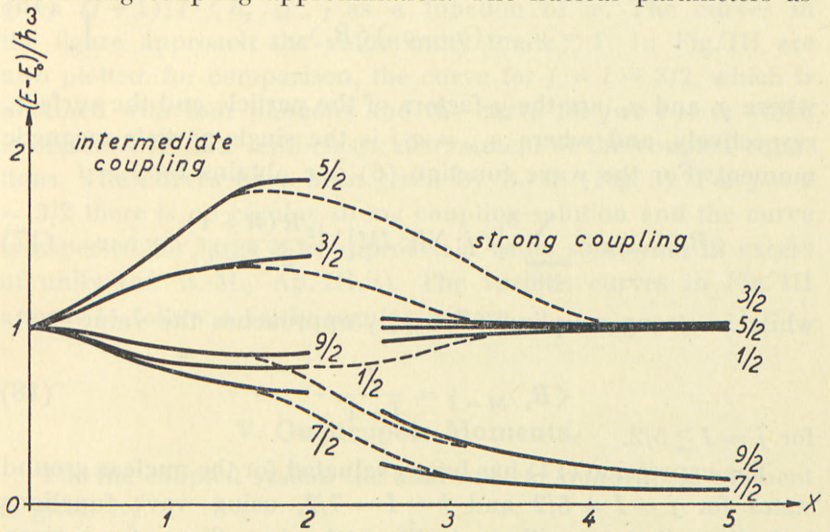

Fig. II. A comparison is made between the energy levels obtained from the intermediate coupling and strong coupling treatments. The curves plotted on the left-hand side are those in Fig. I which have been plotted with respect to the ground state energy. On the right-hand side are those obtained from strong coupling solution where the low lying states may be characterized as rotational and vibrational states. As indicated in the figure, there is a significant gap between the regions of validity of the two methods of treatment, and the interpolations shown by dotted curves cannot claim quantitative validity.

sumed by FORD lead for $A = 100$ and $A = 200$ to the x-values 0.65 and 0.90, respectively, and the results of the intermediate coupling calculations shown in Fig. II should therefore be rather accurate. The level structure obtained shows, however, some essential deviations from that given by FORD. The energy values listed by FORD are too small by more than a factor of two. Apart from the three lowest states, the strong coupling approximation also leads to a different level order from that obtained by the proper intermediate coupling calculations. The conclusion that a single particle configuration does not lead to a strong coupling situation was also drawn by FORD from other arguments.

[1] K.W. FORD, Phys. Rev. 90, 29, 1953.

IV. Magnetic Moments.

For the coupled system consisting of a single particle and the nuclear surface oscillations, the magnetic moment is given by

$$\left.\begin{aligned}\mu &= \langle g_j j_z + g_R R_z \rangle_{M=I} \\ &= \mu_{sp} - (g_j - g_R)\langle R_z \rangle_{M=I},\end{aligned}\right\} \quad (16)$$

where g_j and g_R are the g-factors of the particle and the surface, respectively, and where $\mu_{sp} = g_j j$ is the single-particle magnetic moment. For the wave function (6) one obtains for $j = I$

$$\langle R_z \rangle_{M=I} = \sum_{NR}' |\langle j; NR; IM | \rangle|^2 \frac{R(R+1)}{2(I+1)} \cdots\cdots \quad (17)$$

while in strong coupling $\langle R_z \rangle_{M=I}$ approaches the value

$$\langle R_z \rangle_{M=I} = \frac{I}{I+1} \cdots\cdots \quad (18)$$

for $j = I \geq 5/2$.

The expression (17) has been evaluated for the nuclear ground states for $j = I = 5/2$ and $j = I = 7/2$, using wave functions including all states with up to three phonons. Since, for a given value of x the number of phonons present in the coupled system

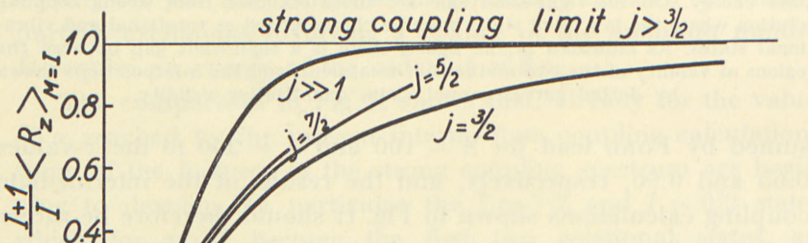

Fig. III. The figure illustrates the transfer of angular momentum from the particle to the surface oscillators as a function of the coupling strength. The curves for $j = I = 5/2$ and $j = I = 7/2$ have been obtained by using wave functions including all unperturbed states with up to three phonons. For comparison, the curves obtained by B.-M. for $j = I = 3/2$ and $j = I \gg 1$ have also been given (cf. their Fig. 10).

increases with j, the three phonon approximation begins to break down for lower values of x in the case of $j = I = 7/2$ than for $j = I = 5/2$. In the former case, the results are considered to be reliable only for $x < 1$.

The results of the calculations are shown in Fig. III which gives $(I+1)/I \cdot \langle R_z \rangle_{M=I}$ as a function of x. The curves in the figure approach the value unity for $x \gg 1$. In Fig. III are also plotted, for comparison, the curve for $j = I = 3/2$, which is obtained with four phonons and the curve for $j = I \gg 1$, which is obtained from a semi-classical treatment of the coupled equations. The curves have been given by B.-M. (Fig. 5). For $j = I = 3/2$ there is no regular strong coupling solution and the curve is expected for large x to approach a value somewhat in excess of unity (cf. B.-M., Ap. III.ii). The various curves in Fig. III appear to follow a fairly regular pattern.

V. Quadrupole Moments.

For the coupled system the total nuclear quadrupole moment is given by

$$Q = Q_p +_s Q \cdots\cdots, \tag{19}$$

where Q_p is the contribution of the particle and Q_s is that of the surface deformation. It is expected that, in general, the latter term contributes the main part of the total Q.

The quadrupole moment Q_s for a uniformly charged nucleus is given by

$$Q_s = \frac{3\,ZR_0^2}{\sqrt{5\pi}} \cdot \langle \alpha_{20} \rangle_{M=I} \cdots\cdots, \tag{20}$$

where Z is the nuclear charge and R_0 the mean radius of the nucleus. Since the particle-surface interaction energy is a linear expression in $\alpha_{2\mu}$, the matrix element in (20) may be conveniently evaluated by using the Schrödinger wave equation*; one then obtains

$$\langle \alpha_{20} \rangle_{M=I} = -\frac{\sqrt{5\pi}}{5} \cdot \frac{8\,I}{(2I+3)} \cdot \frac{\hbar\omega}{k} \cdot \sum_{NR} \left| \langle j; NR; IM | \rangle \right|^2 (N + |\varepsilon|) \cdots\cdots \tag{21}$$

* I am indebted to Dr. K. Alder for suggesting this method.

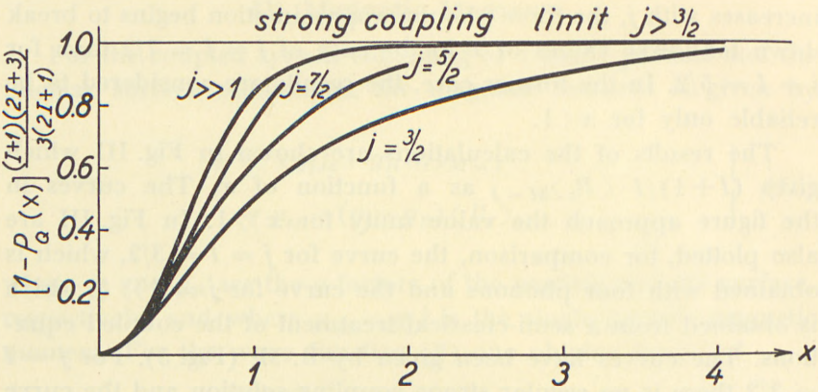

Fig. IV. The figure shows the gradual development of the projection factor for the quadrupole moment in the coupled system as a function of the coupling strength. The curves for $j = I = 5/2$ and $j = I = 7/2$ have been obtained by using wave functions including all unperturbed states with up to three phonons. For comparison, the curves obtained by B.-M. for $j = I = 3/2$ and $j = I \gg 1$ have also been given (cf. their Fig. 10).

It is convenient to write the collective part of the nuclear quadrupole moment in the form (cf. B.-M., Chapter V)

$$Q_s = Q_0 \times P_Q(x) \cdots\cdots\cdots, \tag{22}$$

where

$$Q_0 = -\frac{3\,ZR_0^2}{4\pi}\cdot\frac{k}{C}\cdot\frac{(2I-1)}{2(I+1)}, \tag{23}$$

and where $P_Q(x)$ represents a projection factor which is equal to unity for $x \ll 1$ and approaches for $x \gg 1$, the value

$$P_Q(x) = \frac{I(2I-1)}{(I+1)(2I+3)}\cdots\cdots \tag{24}$$

In strong coupling, where the nucleus performs small vibrations about a certain equilibrium deformation, the quantity Q_0 represents the instrinsic nuclear quadrupole moment measured with respect to the axis of the deformed nucleus.

The transition from weak to strong coupling may be conveniently described in terms of the gradual development of the projection factor (24). For the intermediate coupling wave function, one obtains from (20), (21), (22), and (23)

$$P_Q(x) = \frac{1}{x^2} \cdot \frac{4(I+1)}{(2I+3)(2I-1)} \cdot \sum_{NR}{}' \left| \langle j; NR; IM | \rangle \right|^2 (N + |\varepsilon|) \cdots\cdots (25)$$

This expression has been evaluated for the states $j = I = 5/2$ and $j = I = 7/2$, for the wave function including all possible states with up to three phonons, and the results are given in Fig. IV. In this figure are also shown the intermediate coupling projection factors obtained by B.-M. (cf. their fig. 10) for the states $j = I = 3/2$ and $j = I \gg 1$. The present results tend to confirm the interpolated values of $P_Q(x)$ employed by B.-M. in their analysis of empirical quadrupole moments (cf. Table IX of this reference).

I wish to express my most sincere gratitude to Professor NIELS BOHR for the privilege of working in his Institute and for his kind interest in this problem. I am also greatly indebted to Drs. K. ALDER, A. BOHR, and B. MOTTELSON for many helpful discussions and suggestions.

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